J. Differential Equations 252 (2012) 1748–1775

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

A pde approach to small stochastic perturbations of Hamiltonian flows

Hitoshi Ishii [a,*,1], Panagiotis E. Souganidis [b,2]

[a] *Faculty of Education and Integrated Arts and Sciences, Waseda University, Nishi-Waseda, Shinjuku, Tokyo 169-8050, Japan*
[b] *Department of Mathematics, The University of Chicago, 5734 S. University Avenue, Chicago, IL 60657, USA*

A R T I C L E   I N F O

Article history:
Received 21 March 2011
Revised 13 August 2011
Available online 13 September 2011

MSC:
primary 35B40
secondary 35J25, 37J25, 37J99

Keywords:
Hamiltonian flows
Stochastic perturbations
Averaging
Pde approach

A B S T R A C T

In this note we present a unified approach, based on pde methods, for the study of averaging principles for (small) stochastic perturbations of Hamiltonian flows in two space dimensions. Such problems were introduced by Freidlin and Wentzell and have been the subject of extensive study in the last few years using probabilistic arguments. When the Hamiltonian flow has critical points, it exhibits complicated behavior near the critical points under a small stochastic perturbation. Asymptotically the slow (averaged) motion takes place on a graph. The issues are to identify both the equations on the sides and the boundary conditions at the vertices of the graph. Our approach is very general and applies also to degenerate anisotropic elliptic operators which could not be considered using the previous methodology.

1. Introduction

In this note we present a unified approach, based on pde methods, for the study of averaging principles for (small) stochastic perturbations of Hamiltonian flows in two space dimensions. Such problems were introduced by Freidlin and Wentzell and have been the subject of extensive study in the last few years. When the Hamiltonian flow has critical points, it exhibits complicated behavior near the critical points under a small stochastic perturbation. Asymptotically the slow (averaged) motion takes place on a graph. The issues are to identify both the equations on the sides and the boundary conditions at the vertices of the graph. In their original work Freidlin and Wentzell [6], us-

* Corresponding author.
E-mail addresses: hitoshi.ishii@waseda.jp (H. Ishii), souganidis@math.uchicago.edu (P.E. Souganidis).

[1] Partially supported by the KAKENHI (Nos. 20340019, 21340032, 21224001, 23340028, 23244015), JSPS.
[2] Partially supported by the National Science Foundation.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.08.036

ing probabilistic techniques, considered perturbations by Brownian motions, while later Freidlin and Weber [3] studied, combining probabilistic and analytic techniques based on hypoelliptic operators, a special degenerate case. In [4] the authors considered a Hamiltonian system of d degrees of freedom, with $d > 1$, and so, the phase space is $2d$-dimensional. We also refer to [5] for some related work. More recently Sowers [8,7] revisited the nondegenerate problem and constructed what amounts to approximate correctors for the averaging problem. Although natural, finding such correctors involves serious technical difficulties near the critical points.

In this note we consider anisotropic, possibly degenerate perturbations, thus generalizing significantly the previously known results. Using entirely pde-techniques, we provide a considerably simpler and unified approach. After the statement of the problem we explain our strategy and discuss the new ideas we are introducing here.

We begin by describing the general setting and introducing the necessary material to state the asymptotic problem we are interested in.

We are given a Hamiltonian function

$$H \in C^4(\mathbb{R}^2) \quad \text{such that} \quad \lim_{|x|\to\infty} H(x) = \infty, \tag{1.1}$$

with exactly three nondegenerate critical points z_1, z_2 and z_3. Although it is possible to consider more critical points, to keep the presentation simpler, here we restrict to the case of only three. More precisely, we assume that

$$\begin{cases} \text{there exist } z_1, z_2, z_3 \in \mathbb{R}^2 \text{ such that} \\ DH(z_1) = DH(z_2) = DH(z_3) = 0 \quad \text{and} \quad DH(z) \neq 0 \quad \text{in } \mathbb{R}^2 \setminus \{z_1, z_2, z_3\}, \\ \max\big(H(z_1), H(z_3)\big) < H(z_2), \quad \text{and} \\ \text{the matrices } D^2H(z_1) \text{ and } D^2H(z_3) \text{ are positive definite} \quad \text{and} \quad \det D^2H(z_2) < 0, \end{cases} \tag{1.2}$$

and, to simplify the notation, henceforth we choose

$$z_2 = 0 \quad \text{and} \quad H(0) = 0.$$

It follows from Morse theory (see [1]) that, for any $h > 0$, the open set $\{x \in \mathbb{R}^2\colon\ H(x) < h\}$ is connected and the open set $\{x \in \mathbb{R}^2\colon\ H(x) < 0\}$ has exactly two connected components D_1 and D_3 such that $z_1 \in D_1$ and $z_3 \in D_3$.

Next we choose $h_1, h_2, h_3 \in \mathbb{R}$ such that

$$H(z_1) < h_1 < 0, \qquad 0 = H(z_2) < h_2 \quad \text{and} \quad H(z_3) < h_3 < 0,$$

we consider the open sets

$$\Omega_2 = \big\{x \in \mathbb{R}^2\colon\ 0 < H(x) < h_2\big\}, \quad \text{and,} \quad \text{for } i \in \{1,3\}, \quad \Omega_i = \big\{x \in D_i\colon\ h_i < H(x) < 0\big\},$$

their "outer" boundaries

$$\partial_{\text{out}}\Omega_i = \big\{x \in \bar{\Omega}_i\colon\ H(x) = h_i\big\},$$

as well as the intervals

$$J_2 = (0, h_2) \quad \text{and,} \quad \text{for } i \in \{1,3\}, \quad J_i = (h_i, 0),$$

and, finally, for $i \in \{1,2,3\}$ and $h \in \bar{J}_i$, the "loops"

$$c_i(h) = \big\{x \in \bar{\Omega}_i\colon\ H(x) = h\big\}.$$

The pde we study is set in the connected (a simple argument justifies the last observation) set

$$\Omega = \{x \in \mathbb{R}^2:\ H(x) = 0\} \cup \left(\bigcup_{i=1}^{3} \Omega_i\right),$$

with boundary

$$\partial\Omega = \partial_{\mathrm{out}}\Omega_1 \cup \partial_{\mathrm{out}}\Omega_2 \cup \partial_{\mathrm{out}}\Omega_3.$$

Finally, hence to forth, we write

$$b = \bar{D}H = (H_{x_2}, -H_{x_1}), \tag{1.3}$$

where the subscript x_i indicates the differentiation with respect to the variable x_i.

The problem we are considering here is the asymptotic behavior, as $\varepsilon \to 0$, of the solution u^ε of the boundary value problem

$$\begin{cases} -\operatorname{div}(ADu^\varepsilon) - (b_0 + \varepsilon^{-1}b) \cdot Du^\varepsilon = g & \text{in } \Omega, \\ u^\varepsilon = \rho^\varepsilon \quad \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

We may select here the function b_0 so that $\operatorname{div}(ADu^\varepsilon) + b_0 \cdot Du^\varepsilon = \operatorname{tr} AD^2u^\varepsilon$. Then the behavior of the solutions to (1.4) is related (we refer to [6] for some details) to the asymptotic behavior of the stochastic differential equation

$$dX^\varepsilon = \varepsilon^{-1}\bar{D}H(X^\varepsilon)\,dt + \sqrt{2}\sigma(X^\varepsilon)\,dW,$$

where the matrix σ is the square root of A and W is a Brownian motion in $\mathbb{R}^2$.

We do not know if, in the generality described below, problem (1.4) has a unique solution or not. In this note we do not address this issue but rather we concentrate on the asymptotic analysis.

We assume that

$$\begin{cases} \rho^\varepsilon \in C(\partial\Omega) \text{ and, for each } i \in \{1,2,3\}, \text{ there exists a constant } d_i \\ \text{such that, in the limit } \varepsilon \to 0,\ \rho^\varepsilon \to d_i \text{ uniformly on } \partial_{\mathrm{out}}\Omega_i, \end{cases} \tag{1.5}$$

$$A(x) = (a_{ij})_{1 \leqslant i,j \leqslant 2} \text{ is a smooth, symmetric, nonnegative matrix,} \tag{1.6}$$

$$b_0 \text{ is a smooth vector field,} \tag{1.7}$$

$$\begin{cases} \text{for } i \in \{1,2,3\} \text{ and } h \in \bar{J}_i \setminus \{0\}, \text{ there exists } x_{ih} \in c_i(h) \text{ such that} \\ A(x_{ih})DH(x_{ih}) \cdot DH(x_{ih}) > 0, \end{cases} \tag{1.8}$$

and

$$\begin{cases} \text{in a local orientation-preserving coordinate system at the origin,} \\ \text{where } D^2H(0) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right),\ a_{11}(0) > 0 \text{ and } a_{22}(0) > 0. \end{cases} \tag{1.9}$$

A change of variables is orientation-preserving if its Jacobian is everywhere positive. A coordinate system is orientation-preserving if it is obtained from the original coordinate system by an orientation-preserving change of variables. We remark (see Appendix A) that the form of (1.4) as well as the conditions (1.8) and (1.9) are invariant under any orientation-preserving change of variables.

Regarding (1.9), we note that the Morse lemma (see [1]) yields, after a C^2-orientation-preserving change of variables which fixes the origin, some $\kappa > 0$ such that

$$H(x_1, x_2) = x_1 x_2 \quad \text{in } S_\kappa = \bigl\{x \in \mathbb{R}^2\colon \max\bigl\{|x_1|, |x_2|\bigr\} \leqslant \kappa\bigr\} \subset \Omega, \tag{1.10}$$

and, in these local coordinates,

$$D^2 H(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For future reference we write

$$|D_0 H| = \bigl(a_{11} H_{x_1}^2 + (a_{12} + a_{21}) H_{x_1} H_{x_2} + a_{22} H_{x_2}^2\bigr)^{1/2} = (ADH \cdot DH)^{1/2},$$

and, for a smooth ϕ,

$$\Delta_0 \phi = \operatorname{div}(AD\phi) = (a_{11}\phi_{x_1} + a_{12}\phi_{x_2})_{x_1} + (a_{21}\phi_{x_1} + a_{22}\phi_{x_2})_{x_2}.$$

To state the result, we need some additional preliminary material. To this end, we consider the initial value problem (Hamiltonian system)

$$\dot{X}(t) = \bar{D}H\bigl(X(t)\bigr) \quad \text{and} \quad X(0) = x \in \mathbb{R}^2, \tag{1.11}$$

which admits a unique global in time solution $X(t, x)$. Note that, in view of (1.1),

$$X, \dot{X} \in C^3\bigl(\mathbb{R} \times \mathbb{R}^2; \mathbb{R}^2\bigr) \quad \text{and} \quad H\bigl(X(t, x)\bigr) = H(x) \quad \text{for all } (t, x) \in \mathbb{R} \times \mathbb{R}^2.$$

Fix $i \in \{1, 2, 3\}$ and $h \in \bar{J}_i \setminus \{0\}$. Since $X(\mathbb{R}, x) = \{X(t, x)\colon t \in \mathbb{R}\} \subset c_i(h)$ if $x \in c_i(h)$, and $\bar{D}H(x) \neq 0$ for all $x \in c_i(h)$, it is easily seen that the map $t \mapsto X(t, x)$ is periodic in t for all $x \in c_i(h)$.

It follows from the geometry of the domains Ω_i's that, for any $x \in c_i(h)$ and $h \neq 0$,

$$c_i(h) = X(\mathbb{R}, x), \tag{1.12}$$

and, moreover,

$$\begin{cases} \text{the minimal period } T_i(h) \text{ of } X(\cdot, x) \text{ is independent of } x \in c_i(h) \\ \text{and } 0 < T_i(h) < \infty. \end{cases} \tag{1.13}$$

Throughout the paper, for $i = 1, 3$, we fix $p_i \in c_i(0) \setminus \{0\}$, we denote by $Y_i(h)$ the solution of the initial value problem

$$Y_i'(h) = \frac{DH(Y_i(h))}{|DH(Y_i(h))|^2} \quad \text{and} \quad Y_i(0) = p_i, \tag{1.14}$$

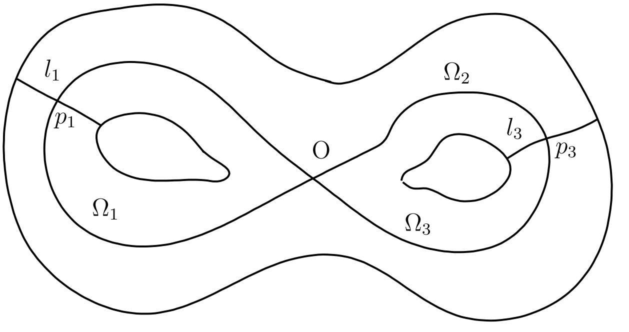
where $Y_i'(h) = dY_i(h)/dh$, and we set

$$l_i = \bigl\{Y_i(t)\colon t \in [h_i, h_2]\bigr\}.$$

It is immediate that

$$H\bigl(Y_i(h)\bigr) = h \quad \text{for all } h \in [h_i, h_2] \quad \text{and} \quad Y_i \in C^3\bigl([h_i, h_2]; \mathbb{R}^2\bigr).$$

To simplify the presentation, we introduce Y_2 and l_2 as well just by setting either $(Y_2, l_2) = (Y_1, l_1)$ or $(Y_2, l_2) = (Y_3, l_3)$.



In the sequel we make several observations and statements which hold true for all $i \in \{1,2,3\}$. To avoid repeating the latter, henceforth, in all statements which hold for all the i's, we will simply write i.

Let $\Phi_i : \mathbb{R} \times \bar{J}_i \to \mathbb{R}^2$ be given by $\Phi_i(t,h) = X(t, Y_i(h))$. It follows that $\Phi_i \in C^3(\mathbb{R} \times \bar{J}_i)$ and, since $H(\Phi_i(t,h)) = h$ for all $(t,h) \in \mathbb{R} \times \bar{J}_i$,

$$\det D\Phi_i = 1 \quad \text{on } \mathbb{R} \times \bar{J}_i. \tag{1.15}$$

The limit of the u^ε's is described by the unique solution $(u_1, u_2, u_3) \in C(\bar{J}_1) \times C(\bar{J}_2) \times C(\bar{J}_3)$ of the boundary value problem

$$\begin{cases} (T_i A_i u_i)'' - (T_i B_i u_i)' + (T_i B_{0i} u_i)' - T_i C_{0i} u_i + T_i \hat{g}_i = 0 \quad \text{in } J_i, \\ \beta_2 u_2'(0) = \sum\limits_{i=1,3} \beta_i u_i'(0), \\ u_1(0) = u_2(0) = u_3(0) \quad \text{and} \quad u_i(h_i) = d_i, \end{cases} \tag{1.16}$$

where, for $h \in J_i$,

$$\begin{cases} A_i(h) = T_i(h)^{-1} \displaystyle\int_0^{T_i(h)} \left| D_0 H\big(\Phi_i(t,h)\big)\right|^2 dt, \\ B_i(h) = T_i(h)^{-1} \displaystyle\int_0^{T_i(h)} \Delta_0 H\big(\Phi_i(t,h)\big)\, dt, \\ B_{0i}(h) = T_i(h)^{-1} \displaystyle\int_0^{T_i(h)} (b_0 \cdot DH)\big(\Phi_i(t,h)\big)\, dt, \\ C_{0i}(h) = T_i(h)^{-1} \displaystyle\int_0^{T_i(h)} \operatorname{div} b_0\big(\Phi_i(t,h)\big)\, dt, \\ \hat{g}_i(h) = T_i(h)^{-1} \displaystyle\int_0^{T_i(h)} g\big(\Phi_i(t,h)\big)\, dt, \\ \beta_i = \lim\limits_{h \to 0+} (A_i T_i)\big((-1)^i h\big). \end{cases} \tag{1.17}$$

As it is shown in Section 2, (1.16) can be rewritten as

$$\begin{cases} A_i u_i'' + (B_i + B_{0i})u_i' + \hat{g}_i = 0 \quad \text{in } J_i, \\ \beta_2 u_2'(0) = \sum\limits_{i=1,3} \beta_i u_i'(0), \\ u_1(0) = u_2(0) = u_3(0) \quad \text{and} \quad u_i(h_i) = d_i. \end{cases} \tag{1.18}$$

The result is:

Theorem 1.1. *Assume* (1.1), (1.2), (1.5), (1.6), (1.7), (1.8), (1.9), *and let* $u^\varepsilon \in C^2(\bar{\Omega})$ *and* $(u_1, u_2, u_3) \in (C^1(\bar{J}_1) \cap C^2(J_1)) \times (C^1(\bar{J}_2) \cap C^2(J_2)) \times (C^1(\bar{J}_3) \cap C^2(J_3))$ *be a solution of* (1.4) *and the unique solution of* (1.16) *respectively. Then, as* $\varepsilon \to 0$,

$$u^\varepsilon \to u_i \circ H \quad \textit{uniformly on } \bar{\Omega}_i. \tag{1.19}$$

This theorem was proved by Freidlin and Wentzell [6] for the Laplacian, i.e., $A = I$, and with slightly more general Hamiltonian H. (Note that our restrictions on H are motivated by the desire to keep the presentation simpler.) Freidlin and Weber [3] studied, using different techniques, a very special degenerate operator, namely $\Delta_0 \phi = \phi_{x_2 x_2}$ and a particular H. Finally Sowers [8,7] considered extensions of [6] and constructed what amounts to approximate correctors.

As remarked earlier here we prove a more general result and provide a unified approach based entirely on pde methods. Our proof not only is simpler than the earlier ones but also introduces several new ideas.

Next we outline some of the key points/steps of the paper. We begin with (1.16). The fact that any uniform in $\bar{\Omega}$ limit of the u^ε's is a function of H is due to the presence of the ε^{-1} factor in front of the b in (1.4). The specific form of (1.16) follows from the above observation and a more or less standard averaging argument. The condition at the vertex is a consequence of simple integration by parts given that the u^ε's solve (1.4) in all of Ω. The heart of the argument is therefore to establish the uniform convergence. This requires uniform in ε estimates, a delicate issue in view of the linearity of the equation and the fact that the matrix A may be degenerate. The former affects possible L^∞-bounds while the second comes in when trying to obtain uniform gradient bounds. In the paper we obtain the L^∞-bounds in an indirect way. First we prove the result assuming such bounds and then we use a classical blow-up argument to obtain the sup-estimates. Assuming the latter we use standard arguments from the theory of viscosity solutions and the periodicity along the trajectories of the Hamiltonian system (for $h \neq 0$) to prove that the largest and smallest possible limits of the u^ε's are solutions of the (1.16) away from the vertex. A key step here is to use (1.8) to prove a local, uniform in ε, L^2-estimate for the derivative of the u^ε's in a direction e. To find this direction we think that $-b = -\bar{D}H$ has the direction of time, and, for $x \in \{|DH| \neq 0\}$, we choose a unit vector $e \neq 0$ so that e and $-b(x)$ span $\mathbb{R}^2$. This enables us to show that, along subsequences, the u^ε's converge in $\{|DH| \neq 0\}$ to a solution of the (1.16) away from the vertex. To conclude we need to prove the convergence on $\bar{\Omega}$. For this we construct appropriate inner and outer barriers that control the behavior near the origin and $\partial\Omega$.

We comment on higher dimensional analogues obtained in [4], where the authors considered a Hamiltonian system of d degrees of freedom, with $d > 1$. In this situation the dynamics are not enough to yield averaging over on each connected component of the level sets of the Hamiltonian except in a very ergodic situation. Indeed, in [4] the driving force for averaging in each level set $M_c = \{x \in \mathbb{R}^{2d}\colon\ H(x) = c\}$, with $c \in \mathbb{R}$ is the sum of the generators of a nondegenerate diffusion process in M_c and of the Hamiltonian flow. Of such a degenerate stochastic process in $\mathbb{R}^{2d}$, they studied a small, nondegenerate stochastic perturbation. Although we have not worked out the details, it is likely that our pde methods may be adapted to the higher dimensional situation in [4], possibly with a degenerate stochastic perturbation.

The paper is organized as follows. Section 2, which is divided into three parts, is devoted to the analysis of (1.16). In the first part we study some properties of the minimal periods. In the last two parts, we consider the coefficients of the ode in (1.16) and provide the general formula for the solution of (1.16). Section 3 is devoted to the proof of Theorem 1.1. It relies on four results that we formulate as separate theorems. They are: a uniform in ε, L^∞-bound for the u^ε's (Theorem 3.1, proved in Section 3), the convergence along subsequences of the u^ε's on the set $\{|D_0H| > 0\}$ (Theorem 3.2, proved in Section 4 combined with Theorem 3.1), and the existence of outer barriers and inner barriers (Theorem 3.3 and Theorem 3.4 respectively, both proved in Section 5). In the proofs we repeatedly perform orientation-preserving changes of variables. We show in Appendix A that such transformations preserve the general structure of the problem. Finally, in Appendix B, we also formulate as a lemma a simple consequence of the classical Green's theorem that we use several times in the proofs.

Throughout the paper we denote by C positive constants, that may change from line to line and are independent of ε. The latter is always taken to be positive. Moreover we use the term "solution" to mean either a classical (if smooth) or a viscosity (if only continuous) solution.

We conclude with the notation we use in the paper.

Notation. For any $a, b \in \mathbb{R}$, $x = (x_1, x_2) \in \mathbb{R}^2$, $f : V \to \mathbb{R}^k$, $V \subset \mathbb{R}^m$, symmetric matrix A of order k, and family of bounded functions $w_\varepsilon : \Omega \to \mathbb{R}$ we have

$$\begin{cases} |x|_\infty = \max\{|x_1|, |x_2|\}, \\ \|f\|_{\infty,V} = \sup\{|f(x)|:\ x \in V\}, \\ \{f > 0\} = \{x \in V:\ f(x) > 0\} \quad \text{for } k = 1, \\ w^+(x) = \limsup{}^* w^\varepsilon(x) = \limsup\limits_{y\to x,\, \varepsilon\to 0} w^\varepsilon(y), \\ w^-(x) = \liminf{}_* w^\varepsilon(x) = \liminf\limits_{y\to x,\, \varepsilon\to 0} w^\varepsilon(y). \end{cases}$$

2. The limit problem

2.1. Some properties of the minimal period

We study here the regularity and the behavior for small h of the minimal periods T_1, T_2 and T_3. Both are necessary for the regularity of the coefficients of (1.16) as well as some other estimates later in the paper. At first passage the reader may choose to skip the proofs.

We begin with

Lemma 2.1. $T_i \in C^3(\bar{J}_i \setminus \{0\})$.

Proof. Since $Y_i \in C^3(\bar{J}_i)$ is injective, we may choose $\phi \in C^3(\mathbb{R}^2)$ such that, for all $x \in l_i$, $\phi(x) = 0$ and $D\phi(x) \neq 0$.

Set $\psi(t,h) = \phi(X(t, Y_i(h)))$ for $(t,h) \in \mathbb{R} \times \bar{J}_i$, and note that, for all $h \in \bar{J}_i \setminus \{0\}$, $\psi(T_i(h), h) = \phi(Y_i(h)) = 0$. Moreover, since, for any $h \in \bar{J}_i$, the vectors $D\phi(Y_i(h))$ and $\bar{D}H(Y_i(h))$ are parallel to each other, we see that, if $h \in \bar{J}_i \setminus \{0\}$, then

$$\psi_t(T_i(h), h) = D\phi(Y_i(h)) \cdot \dot{X}(T_i(h), Y_i(h)) = D\phi(Y_i(h)) \cdot \bar{D}H(Y_i(h)) \neq 0,$$

and the claim follows from the implicit function theorem. $\square$

The small h behavior of the T_i's is the subject of

Lemma 2.2. *There exists $C > 0$ such that, for all $h \in J_i$,*

$$C^{-1} \log(|h|^{-1} + 2) \leqslant T_i(h) \leqslant C \log(|h|^{-1} + 2). \tag{2.1}$$

Proof. Since the arguments are similar we only prove (2.1) for T_2, which, for notational simplicity, we denote for the rest of the proof by T.

In view of (1.10), we have

$$m = \inf\{\left|DH(x)\right|:\ x \in \Omega,\ \left|H(x)\right| \geqslant \kappa^2/4\} > 0.$$

Let $h \in (0, \kappa^2/4)$, fix $x \in \Omega_2$ so that $h = H(x)$, consider the trajectory $X(t) = X(t, x)$ and observe that $X(\bar{t}) = \sqrt{h}(1,1) \in S_\kappa$ for some $\bar{t} \in [0, T(h))$. Assuming that, after a translation, $X(0) = \sqrt{h}(1,1)$, we find that $X(t) = \sqrt{h}(e^t, e^{-t})$ for all $t \in [0, \tau]$ with $\tau > 0$ given by $\sqrt{h}e^\tau = \kappa$. It is then clear that $\tau < T(h)$ and, hence,

$$T(h) > \log\left(\kappa h^{-1/2}\right) \quad \text{for } 0 < h < \kappa^2/4. \tag{2.2}$$

If $\operatorname{diam}(B)$ denotes the diameter of the set B, we have

$$2\operatorname{diam}\left(c_2(0)\right) \leqslant 2\operatorname{diam}\left(c_2(h)\right) \leqslant \int_0^{T(h)} |\dot{X}|\,dt = \int_0^{T(h)} \left|DH(X)\right| dt \leqslant T(h) \sup_\Omega |DH|,$$

and, thus,

$$T(h) \geqslant 2\left(\sup_\Omega |DH|\right)^{-1} \operatorname{diam}\left(c_2(0)\right),$$

which yields, in view of (2.2), the lower bound for T_2 in (2.1).

Applying Green's theorem, for $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$, we find

$$\int_{\{h<H<h_2\}} \Delta H\,dx = \int_0^{T(h_2)} \left|DH\left(\Phi_2(t, h_2)\right)\right|^2 dt - \int_0^{T(h)} \left|DH\left(X(t)\right)\right|^2 dt.$$

Accordingly, if $h \geqslant \kappa^2/4$,

$$m^2 T(h) \leqslant \int_0^{T(h)} \left|DH\left(X(t)\right)\right|^2 dt \leqslant \int_\Omega |\Delta H|\,dx + \int_0^{T(h_2)} \left|DH\left(\Phi_2(t, h_2)\right)\right|^2 dt,$$

and, hence,

$$T(h) \leqslant m^{-2}\left(\int_\Omega |\Delta H|\,dx + \int_{c_2(h_2)} |DH|\,dl\right). \tag{2.3}$$

On the other hand, if $0 < h < \kappa^2/4$, then assuming, as before, that $X(0) = \sqrt{h}(1,1)$ and $\sqrt{h}e^\tau = \kappa$, we find

$$\int_\Omega |\Delta H|\,dx + \int_{c_2(h_2)} |DH|\,dl \geqslant \int_\tau^{T(h)-\tau} \left|DH\left(X(t)\right)\right|^2 dt \geqslant m^2\left(T(h) - 2\tau\right),$$

and, hence,

$$
\begin{aligned}
T(h) &\leqslant 2\tau + m^{-2}\Bigl(\int_{\Omega} |\Delta H|\,dx + \int_{c_2(h_2)} |DH|\,dl\Bigr) \\
&\leqslant \frac{1}{2}\log\bigl(\kappa^2 h^{-1}\bigr) + m^{-2}\Bigl(\int_{\Omega} |\Delta H|\,dx + \int_{c_2(h_2)} |DH|\,dl\Bigr).
\end{aligned}
$$

Combining the above estimate and (2.3) yields, for some other $C > 0$, the second inequality in (2.1) for T_2. $\square$

2.2. The coefficients of the ode in (1.16)

Here we establish the properties (positivity, regularity as well as what is necessary to show (1.18)) of the coefficients of the ode in (1.16).

Applying Lemma B.1 to

$$
f_1 = a_{11}H_{x_1} + a_{12}H_{x_2} \quad \text{and} \quad f_2 = a_{21}H_{x_1} + a_{22}H_{x_2},
$$

with $\alpha_i = T_i(h)$ for $h \in J_i$, and differentiating the resulting formula with respect to h, we obtain

$$
\begin{aligned}
(T_iA_i)'(h) &= \frac{d}{dh}\int_0^{T_i(h)} \bigl|D_0H \circ \Phi_i(t,h)\bigr|^2\,dt \\
&= \frac{d}{dh}\int_0^{T_i(h)} (f_1H_{x_1} + f_2H_{x_2}) \circ \Phi(t,h)\,dt \\
&= \int_0^{T_i(h)} (f_{1,x_1} + f_{2,x_2}) \circ \Phi_i(t,h)\,dt = \int_0^{T_i(h)} \Delta_0 H \circ \Phi_i(t,h)\,dt \\
&= (T_iB_i)(h),
\end{aligned} \tag{2.4}
$$

and

$$
\begin{aligned}
(T_iB_{0i})'(h) &= \frac{d}{dh}\int_0^{T_i(h)} \bigl|(b_0 \cdot DH) \circ \Phi_i(t,h)\bigr|^2\,dt = \int_0^{T_i(h)} \operatorname{div} b_0 \circ \Phi_i(t,h)\,dt \\
&= (T_iC_{0i})(h).
\end{aligned} \tag{2.5}
$$

We have:

Lemma 2.3. *$A_i, B_i \in C^2(\bar{J}_i \setminus \{0\})$, $A_i > 0$, and (2.4) and (2.5) hold for all $h \in J_i$.*

Proof. The positivity of the A_i's is immediate from (1.17) and (1.8), the claimed regularity follows from the fact that $T_i \in C^3(\bar{J}_i \setminus \{0\})$, and the formulae were derived above. $\square$

Using (2.4) and (2.5) we find that the equation in (1.16) can be rewritten as

$$\big(T_i A_i u_i'\big)' + T_i B_{0i} u_i' + T_i \hat{g}_i = 0 \quad \text{in } J_i, \tag{2.6}$$

or

$$A_i u_i'' + (B_i + B_{0i}) u_i' + \hat{g}_i = 0 \quad \text{in } J_i. \tag{2.7}$$

We have:

Lemma 2.4. *The constants β_i in* (1.17) *are well defined and positive and the functions $T_i A_i$, $(T_i A_i)^{-1}$, $T_i B_{0i}$ and B_{0i}/A_i are uniformly continuous in J_i.*

Proof. It is immediate from (1.17) that A_i, B_i and $\hat{g}_i$ are bounded. Accordingly, in view of Lemma 2.2, we see that $T_i B_i \in L^1(J_i)$ and, since

$$(T_i A_i)(h) = (T_i A_i)(h_i) + \int_{h_i}^{h} (T_i B_i)(\eta)\, d\eta,$$

$T_i A_i$ is uniformly continuous in J_i and the limits $\beta_i = \lim_{h \to 0+} (T_i A_i)((-1)^i h)$ exist.

We focus now on $T_2 A_2$, since the arguments for $T_1 A_1$ and $T_3 A_3$ are similar. In view of (1.9) and (1.10), we may choose (by taking $\kappa > 0$ small enough) constants $0 < a_0 \leqslant a_1 < \infty$ such that, for $x \in S_\kappa$,

$$a_0 \leqslant \min\big\{a_{11}(x), a_{22}(x)\big\} \leqslant \max\big\{a_{11}(x), a_{22}(x)\big\} \leqslant a_1$$

and, hence,

$$\big|a_{12}(x)\big| = \big|a_{21}(x)\big| \leqslant \sqrt{a_{11}(x) a_{22}(x)} \leqslant a_1.$$

Fix $h \in (0, e^{-4a_1/a_0}\kappa^2)$, set $x = \sqrt{h}(1,1) \in V$ and $X(t) = X(t,x)$, and recall that $X(t) = \sqrt{h}(e^t, e^{-t})$ for $|t| \leqslant \tau$, where, as before, $\sqrt{h} e^\tau = \kappa$.

As in the proof of Lemma 2.2, we find that $T_2(h) > \tau$ and, hence,

$$\begin{aligned}\int_0^{T_2(h)} \big|D_0 H\big(X(t)\big)\big|^2\, dt &\geqslant \int_0^{\tau} \big(a_0 X_2(t)^2 - 2a_1\big|X_1(t)X_2(t)\big| + a_0 X_2(t)^2\big)\, dt \\ &= h \int_0^{\tau} \big(a_0\big(e^{2t} + e^{-2t}\big) - 2a_1\big)\, dt > \frac{a_0 h}{2}\big(e^{2\tau} - 1\big) - 2a_1 h\tau.\end{aligned}$$

Noting that

$$2\tau = \log\frac{\kappa^2}{h} \geqslant \frac{4a_1}{a_0} \quad \text{and} \quad e^{2\tau} - 1 > 2\tau + 2\tau^2 > 2\tau^2 \geqslant \frac{8a_1\tau}{a_0},$$

we get

$$(T_2 A_2)(h) \geqslant \frac{a_0 h}{4}\big(e^{2\tau} - 1\big) + 2a_1 h\tau - 2a_1 h\tau = \frac{a_0}{4}\big(\kappa^2 - h\big) \geqslant \frac{a_0}{4}\big(1 - e^{-\frac{4a_1}{a_0}}\big)\kappa^2.$$

Since $T_2A_2 > 0$ in $(0, h_2]$, we conclude that

$$\inf_{h\in J_2} (T_2A_2)(h) > 0,$$

and hence, $\beta_i > 0$, and the function $(T_iA_i)^{-1}$ is uniformly continuous in J_i.

Similarly, we have $T_iC_{0i} \in L^1(J_i)$ and

$$(T_iB_{0i})(h) = (T_iB_{0i})(h_i) + \int_{h_i}^{h} (T_iC_{0i})(\eta)\,d\eta,$$

and therefore, the function T_iB_{0i} is uniformly continuous in J_i.

The last claim is a consequence of the already obtained regularity. □

Set

$$\gamma_i = (T_iB_{0i})(0) = \lim_{h\to 0+} (T_iB_{0i})\big((-1)^i h\big). \tag{2.8}$$

We have:

Lemma 2.5. *$\sum_{i=1}^3 (-1)^i\beta_i = 0$ and $\sum_{i=1}^3 (-1)^i\gamma_i = 0$.*

Proof. Fix $0 < \varepsilon < \min\{(-1)^i h_i\colon\ i = 1, 2, 3\}$, let $\Omega(\varepsilon) = \{x \in \Omega\colon\ |H(x)| < \varepsilon\}$, observe that $\partial\Omega(\varepsilon)$ has three connected components $c_1(-\varepsilon)$, $c_2(\varepsilon)$ and $c_3(-\varepsilon)$, and note that, for $x \in c_2(\varepsilon)$, $DH(x)$ points outward from $\Omega(\varepsilon)$, while, for $i = 1, 3$ and $x \in c_i(-\varepsilon)$, $DH(x)$ points inward to $\Omega(\varepsilon)$.

Using Lemma B.1 with $f_1 = a_{11}H_{x_1} + a_{12}H_{x_2}$ and $f_2 = a_{21}H_{x_1} + a_{22}H_{x_2}$, we find

$$\int_{\Omega(\varepsilon)} \Delta_0 H(x)\,dx = -\sum_{i=1,3} \int_0^{T_i(-\varepsilon)} |D_0H|^2\big(\Phi_i(t, -\varepsilon)\big)\,dt + \int_0^{T_2(\varepsilon)} |D_0H|^2\big(\Phi_2(t, \varepsilon)\big)\,dt.$$

Similarly we have

$$\int_{\Omega(\varepsilon)} \operatorname{div} b_0(x)\,dx = -\sum_{i=1,3} \int_0^{T_i(-\varepsilon)} (b_0 \cdot DH)\big(\Phi_i(t, -\varepsilon)\big)\,dt + \int_0^{T_2(\varepsilon)} (b_0 \cdot DH)\big(\Phi_2(t, \varepsilon)\big)\,dt.$$

Letting $\varepsilon \to 0$ yields the claim. □

2.3. The boundary value problem for the ode

Solutions $u = (u_1, u_2, u_3)$ of the ode (1.18), without the boundary conditions, are given, for some constants C_{ij}, with $i = 1, 2, 3,\ j = 1, 2$, by

$$\begin{aligned} u_i(h) = C_{i1} + C_{i2} \int_0^h (T_iA_i)(\eta)^{-1} e^{-\int_0^\eta B_{0i}(t)A_i(t)^{-1}\,dt}\,d\eta \\ - \int_0^h (T_iA_i)(\eta)^{-1} \int_0^\eta e^{-\int_\xi^\eta B_{0i}(s)A_i(s)^{-1}\,ds} (T_i\hat{g}_i)(\xi)\,d\xi\,d\eta. \end{aligned} \tag{2.9}$$

Using the boundary conditions of (1.18) in (2.9) we find

$$u_i(0) = C_{i1}, \qquad u_i'(0) = \frac{C_{i2}}{(T_i A_i)(0)} = \frac{C_{i2}}{\beta_i} \quad \text{and} \quad u_i(h_i) = C_{i1} + C_{i2} P_i - Q_i, \tag{2.10}$$

where

$$P_i = \int_0^{h_i} (T_i A_i)(\eta)^{-1} e^{-\int_0^\eta B_{0i}(t) A_i(t)^{-1}\,dt}\,d\eta,$$

$$Q_i = \int_0^{h_i} (T_i A_i)(\eta)^{-1} \int_0^\eta e^{-\int_\xi^\eta B_{0i}(s) A_i(s)^{-1}\,ds} (T_i \hat{g}_i)(\xi)\,d\xi\,d\eta. \tag{2.11}$$

The above and the boundary conditions at the vertex in (1.18) lead to the linear system

$$C_{11} = C_{21} = C_{31}, \qquad C_{22} = \sum_{i=1,3} C_{i2} \quad \text{and} \quad C_{i1} + C_{i2} P_i - Q_i = d_i,$$

whose unique solution is given by

$$C_{i1} = \frac{\sum_{i=1}^3 (-1)^i P_i^{-1}(d_i + Q_i)}{\sum_{i=1}^3 (-1)^i P_i^{-1}} \quad \text{and} \quad C_{i2} = P_i^{-1}(d_i + Q_i - C_{i1}). \tag{2.12}$$

3. The proof of the main theorem

We formulate here as theorems the steps, described in the informal discussion at the end of the Introduction, that lead to the proof of Theorem 1.1.

We have:

Theorem 3.1 *(Uniform bound). Assume* (1.1), (1.2), (1.5), (1.6), (1.7), (1.8), (1.9) *and let u^ε be a solution of* (1.4). *There exists $\varepsilon_0 > 0$ such that*

$$\sup_{0<\varepsilon<\varepsilon_0} \|u^\varepsilon\|_{\infty,\Omega} < \infty. \tag{3.1}$$

Theorem 3.1, which is very important for the proof of Theorem 1.1, is proved by a blow-up argument provided that we can first prove it under the additional assumption that (3.1) holds. The proof of the convergence of the u^ε's, if (3.1) holds, consists of three steps which we formulate as separate theorems. The first is to show, that along subsequences, the u^ε's converge, locally uniformly, in $\Omega \setminus \{0\}$. The next two steps entail the construction of appropriate barriers yielding the convergence away from the origin and, finally, on $\bar{\Omega}$.

We have:

Theorem 3.2 *(Precompactness). Assume* (1.1), (1.2), (1.5), (1.6), (1.7), (1.8), (1.9), *and let u^ε be a solution of* (1.4) *and set $N = \{x \in \Omega\colon |D_0 H(x)| > 0\}$. Let $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ be a sequence converging to zero such that $\sup_{j\in\mathbb{N}} \|u^{\varepsilon_j}\|_{\infty,\Omega} < \infty$. Then the family $\{u^{\varepsilon_j}\}$ is precompact in $C(N)$.*

Theorem 3.3 *(Outer barriers). Assume* (1.1), (1.2), (1.5), (1.6), (1.7), (1.8), (1.9), *let* $0 < h_0 < \min_{i=1,2,3} |h_i|$ *and set* $I_i = (h_i, -h_0)$ *if* $i = 1, 3$ *and* $I_2 = (h_0, h_2)$. *There exist* $\varepsilon_0 \in (0, 1)$ *and families* $\{w_i^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)} \subset C^2(\bar{\Omega}_i \cap \{|H| \geqslant h_0\})$, *such that*

$$-\big(\Delta_0 + \big(b_0 + \varepsilon^{-1} b\big) \cdot D\big) w_i^\varepsilon \leqslant -1 \quad \text{in } \Omega_i \cap \big\{|H| > h_0\big\},$$
$$w_i^\varepsilon \leqslant -1 \quad \text{on } \Omega_i \cap \big\{|H| = h_0\big\},$$

and, as $\varepsilon \to 0$, *the* w_i^ε*'s converge uniformly to some* $w_i \in C(\bar{\Omega}_i \cap \{(-1)^i H \geqslant h_0\})$ *and* $w_i^\varepsilon \to d_i$ *uniformly on* $\partial_{\mathrm{out}} \Omega_i$.

Theorem 3.4 *(Inner barriers). Assume* (1.1), (1.2), (1.5), (1.6), (1.7), (1.8) *and* (1.9). *There exist* $\varepsilon_0 \in (0, 1)$, *a neighborhood* $V \subset \Omega$ *of the origin and a family* $\{v^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)} \subset C^2(V)$ *such that*

$$-\big(\Delta_0 + \big(b_0 + \varepsilon^{-1} b\big) \cdot D\big) v^\varepsilon \leqslant -1 \quad \text{in } V,$$

and, as $\varepsilon \to 0$, $v^\varepsilon \to 0$ *uniformly on* V.

Assuming temporarily Theorems 3.1, 3.2, 3.3 and 3.4, we continue with the

Proof of Theorem 1.1. Theorems 3.1 and 3.2 yield the existence of sequences $\varepsilon_j \to 0$ along which the u^{ε_j}'s converge locally uniformly in N.

In order to show the convergence of the whole family u^ε in $\bar{\Omega}$, it is enough to prove that the u^{ε_j}'s converge uniformly in $\bar{\Omega}_i$ to $u_i \circ H$ – recall that (u_1, u_2, u_3) is the unique solution of (1.16).

We introduce next the classical half-relaxed limits (see [2])

$$u^+ = \limsup{}^* u^{\varepsilon_j} \quad \text{and} \quad u^- = \liminf{}_* u^{\varepsilon_j},$$

which, in view of Theorem 3.1, are well defined and bounded on $\bar{\Omega}$. The aim is thus to show that $u^+ = u^- = u_i \circ H$ on $\bar{\Omega}_i$.

The first step is to prove that, for each i, there exists some $v_i \in C(J_i)$ such that $u^+ = u^- = v_i \circ H$ in Ω_i.

Noting that the theory of viscosity solutions yields

$$-b \cdot Du^+ \leqslant 0 \quad \text{and} \quad -b \cdot Du^- \geqslant 0 \quad \text{in } \Omega,$$

it follows that u^+ and u^- are respectively nondecreasing and nonincreasing along the curve $(X(t, x))_{t\in\mathbb{R}}$ given by (1.11).

Next fix $i \in \{1, 2, 3\}$ and $x \in \Omega_i$ and set $h = H(x)$. The monotonicity of u^+ along the curve $(X(t, x))_{t\in\mathbb{R}}$ yields, for all $t \in [0, T_i(h)]$,

$$u^+(x) = u^+\big(X\big(T_i(h), x\big)\big) \geqslant u^+\big(X(t, x)\big) \geqslant u^+\big(X(0, x)\big) = u^+(x),$$

i.e., u^+ is constant on the loop $c_i(h)$. Similarly, we find that u^- is constant on the loop $c_i(h)$ as well.

Since, in view of (1.8), the loop $c_i(h)$ intersects N, and, by the choice of the ε_j's, $u^+ = u^-$ in $c_i(h) \cap N$ (recall that the u^{ε_j}'s converge in N), we finally find that, for some constant $v_i(h)$ depending on i and h,

$$u^+ = u^- = v_i(h) \quad \text{on } c_i(h). \tag{3.2}$$

In particular, $u^+ = u^-$ in $\Omega \setminus \{H \neq 0\}$, which implies that $u^+ = u^- \in C(\Omega \setminus \{H \neq 0\})$, and, hence, $v_i \in C(J_i)$.

The next step is to establish that $u^+ = u^-$ in $\{H = 0\} \setminus \{0\}$. We assume in the rest of proof that (1.10) holds. Then we have that

$$\{H = 0\} \cap S_\kappa = \{x \in S_\kappa\colon x_1 = 0\} \cup \{x \in S_\kappa\colon x_2 = 0\},$$

$$\{H = 0\} = c_2(0) = c_1(0) \cup c_3(0),$$

and, for $x \in S_\kappa \setminus \{0\}$,

$$\bigl|D_0 H(0, x_2)\bigr|^2 = a_{22}(0, x_2)x_2^2 \quad \text{and} \quad \bigl|D_0 H(x_1, 0)\bigr|^2 = a_{11}(x_1, 0)x_1^2.$$

In view of (1.9), we may also choose $\kappa > 0$ small enough so that $|D_0 H|^2 > 0$ in $\{H = 0\} \cap S_\kappa \setminus \{0\}$. It follows that $\{H = 0\} \cap S_\kappa \setminus \{0\} \subset N$ and, hence,

$$u^+ = u^- \quad \text{in } \{H = 0\} \cap S_\kappa \setminus \{0\}.$$

Next we fix $i \in \{1, 3\}$ and $x \in c_i(0) \setminus \{0\}$, and note that there exist $y, z \in c_i(0) \cap S_\kappa$ and $s < 0 < t$ such that $X(s, x) = y$ and $X(t, x) = z$. Using, as above, the monotonicity of $u^\pm$ along the curves $(X(t, x))_{t\in\mathbb{R}}$, we conclude that $u^+ = u^- = v_i(0)$ on $c_i(0) \setminus \{0\}$ for some constant $v_i(0)$. Moreover, we see that $v_i(0) = \lim_{h\to 0+} v_i(h) = \lim_{h\to 0-} v_i(h)$. In particular, setting $v_2(0) = \lim_{h\to 0+} v_2(h)$, we find that

$$v_1(0) = v_2(0) = v_3(0) \quad \text{and} \quad v_i \in C\bigl(J_i \cup \{0\}\bigr).$$

Now we prove that $u^+(0) = u^-(0)$. Observe that, in view of Theorem 3.4, there exist $\varepsilon_0 > 0$, a neighborhood V of the origin, and a family $\{v^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)} \subset C^2(V)$ such that

$$-\bigl(\Delta_0 + \bigl(b_0 + \varepsilon^{-1} b\bigr) \cdot D\bigr) v^\varepsilon \leqslant -1 \quad \text{in } V \quad \text{and} \quad \lim_{\varepsilon\to 0} \bigl\| v^\varepsilon \bigr\|_{\infty,V} = 0.$$

By choosing $\kappa > 0$ even smaller, if needed, we may assume that $S_\kappa \subset V$. For $\delta > 0$, we set

$$S_{\kappa,\delta} = \bigl\{x \in S_\kappa\colon \bigl|H(x)\bigr| \leqslant \delta\bigr\} \quad \text{and} \quad e_j(\kappa, \delta) = \max\bigl\{\bigl|u^{\varepsilon_j}(x) - v_1(0)\bigr|\colon x \in \partial S_{\kappa,\delta}\bigr\},$$

and observe that, since $v_1(0) = v_2(0) = v_3(0)$, for each $i \in \{1, 2, 3\}$,

$$\lim_{\delta\to 0} \lim_{j\to\infty} e_j(\kappa, \delta) = \lim_{\delta\to 0} \max\bigl\{\bigl|v_i(h) - v_i(0)\bigr|\colon 0 \leqslant (-1)^i h \leqslant \delta\bigr\} = 0.$$

Set

$$f_j = v_1(0) - e_j(\kappa, \delta) - \bigl(\|g\|_{\infty,\Omega} + 1\bigr)\bigl(\bigl\| v^{\varepsilon_j} \bigr\|_{\infty,V} + v^{\varepsilon_j}\bigr) \quad \text{in } S_{\kappa,\delta},$$

and note that

$$-\bigl(\Delta_0 + \bigl(b_0 + \varepsilon^{-1} b\bigr) \cdot D\bigr) f_j \leqslant -\|g\|_{\infty,\Omega} - 1 \leqslant g - 1 \quad \text{in } S_{\kappa,\delta},$$
$$u^{\varepsilon_j} \geqslant f_j \quad \text{on } \partial S_{\kappa,\delta}.$$

The maximum principle implies that $u^{\varepsilon_j} \geqslant f_j$ on $S_{\kappa,\delta}$, and, hence, after sending first $j \to \infty$ and then $\delta \to 0$, we get $u^-(0) \geqslant v_1(0)$.

A similar argument with f_j replaced by the function

$$f_j = v_1(0) + e_j(\kappa,\delta) + \left(\|g\|_{\infty,\Omega} + 1\right)\left(\left\|v^{\varepsilon_j}\right\|_{\infty,V} - v^{\varepsilon_j}\right) \quad \text{for } x \in S_{\kappa,\delta},$$

yields $u^+(0) \leqslant v_1(0)$ and, thus, $u^+(0) = u^-(0) = v_i(0)$ for $i \in \{1,2,3\}$.

Fix $h_0 \in (0, \min_{i=1,2,3} |h_i|)$ and observe that, in view of Theorem 3.1, there exist $\varepsilon_0, M > 0$ so that $M > \|g\|_{\infty,\bar{\Omega}} + \sup_{\varepsilon\in(0,\varepsilon_0)} \|u^\varepsilon\|_{\infty,\Omega}$.

Replacing, if needed, ε_0 by a smaller positive number, we recall that Theorem 3.3 yields a family $\{w^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)} \subset C^2(\bar{\Omega}_i \cap \{|H| \geqslant h_0\})$ which converges uniformly in $\Omega \cap \{|h| \geqslant h_0\}$.

In addition,

$$-\left(\Delta_0 + \left(b_0 + \varepsilon^{-1} b\right)\cdot D\right) w^\varepsilon \leqslant -1 \quad \text{in } \Omega \cap \left\{|H| > h_0\right\} \quad \text{and} \quad w^\varepsilon \leqslant -1 \quad \text{on } \Omega_i \cap \left\{|H| = h_0\right\},$$

and

$$\lim_{\varepsilon\to 0} w^\varepsilon = M^{-1} d_i \quad \text{on } \partial_{\mathrm{out}}\Omega_i.$$

If

$$f^\varepsilon = M w^\varepsilon - \max\left\{\left|\rho^\varepsilon(x) - d_i\right| \colon x \in \partial_{\mathrm{out}}\Omega_i,\ i = 1,2,3\right\},$$

then

$$-\left(\Delta_0 + \left(b_0 + \varepsilon^{-1} b\right)\cdot D\right) f^\varepsilon \leqslant -M < g \quad \text{in } \Omega \cap \left\{|H| > h_0\right\},$$
$$f^\varepsilon \leqslant u^\varepsilon \quad \text{on } \partial\left(\Omega \cap \left\{|H| > h_0\right\}\right) \quad \text{and} \quad \lim_{\varepsilon\to 0} f^\varepsilon = d_i \quad \text{uniformly on } \partial_{\mathrm{out}}\Omega_i.$$

Since, by comparison, $f^\varepsilon \leqslant u^\varepsilon$ on $\bar{\Omega} \cap \{|H| > h_0\}$, for each $i \in \{1,2,3\}$, we find that $d_i \leqslant u^-$ on $\partial_{\mathrm{out}}\Omega_i$, and, by a similar argument, $d_i \geqslant u^+$ on $\partial_{\mathrm{out}}\Omega_i$. Hence, $u^- = u^+ = d_i$ on $\partial_{\mathrm{out}}\Omega_i$.

Since $u^+ = u^-$ everywhere, it is now easy to see that, as $j \to \infty$,

$$u^{\varepsilon_j} \to v_i \circ H \quad \text{uniformly on } \bar{\Omega}_i.$$

The last step is to identify (v_1, v_2, v_3) as the unique solution of (1.16), and, hence, conclude that, as $\varepsilon \to 0$, the u^ε's converge to $u_i \circ H$ uniformly in $\bar{\Omega}$.

To this end, fix $i \in \{1,2,3\}$ and $\phi \in C_0^2(J_i)$. Integrating (1.4) by parts, we find

$$\int_{\Omega_i} \left\{u^\varepsilon\left(\phi'' \circ H |D_0 H|^2 + \phi' \circ H(\Delta_0 H - b_0 \cdot DH) - \operatorname{div} b_0 \phi \circ H\right) + g\phi \circ H\right\} dx = 0.$$

Taking $\varepsilon = \varepsilon_j$ and sending $j \to \infty$, we get

$$\int_{J_i} T_i(h)\left\{\left(A_i v_i \phi'' + (B_i - B_{0i}) v_i \phi'\right) + (\hat{g}_i - C_{0i} v_i)\phi\right\} dh = 0. \tag{3.3}$$

Since ϕ is arbitrary, in view of (2.5) and (2.6), (3.3) gives

$$A_i v_i'' + (B_i + B_{0i}) v_i' + \hat{g}_i = 0 \quad \text{in } J_i.$$

Next fix $\phi \in C_0^2(J)$ with $J = (\max\{h_1, h_3\}, h_2)$, and observe, as above, that

$$\int_\Omega \big\{u^\varepsilon\big(\phi'' \circ H|D_0H|^2 + \phi' \circ H(\Delta_0 H - b_0 \cdot DH) - \operatorname{div} b_0\phi \circ H\big) + g\phi \circ H\big\}\,dx = 0,$$

and, therefore,

$$\begin{aligned} 0 = &\int_0^{h_2} T_2\big\{\big(A_2 v_2 \phi'' + (B_2 - B_{02}) v_2 \phi'\big) + (\hat{g}_2 - C_{02} v_2)\phi\big\}\,dh \\ &+ \sum_{i=1,3} \int_{h_i}^0 T_i\big\{\big(A_i v_i \phi'' + (B_i - B_{0i}) v_i \phi'\big) + (\hat{g}_i - C_{0i} v_i)\phi\big\}\,dh. \end{aligned} \tag{3.4}$$

Since

$$\big(T_i A_i v_i \phi' - T_i A_i v_i' \phi\big)' = (T_i A_i)' v_i \phi' + T_i A_i \phi'' - \big(T_i A_i v_i'\big)'\phi,$$

and

$$(T_i B_{0i} v_i \phi)' = T_i B_{0i} v_i \phi' + (T_i B_{0i})' v_i \phi + T_i B_{0i} v_i' \phi,$$

integrating by parts, with $0 < \delta < h_2$, and using (2.4) and (2.5) we find

$$\begin{aligned} &-\big(T_2 A_2 v_2 \phi' - T_2 A_2 v_2' \phi - T_2 B_{02} v_2 \phi\big)(\delta) \\ &= \int_\delta^{h_2} \big\{(T_2 A_2)' v_2 \phi' + T_2 A_2 \phi'' - \big(T_2 A_2 v_2'\big)'\phi - \big(T_2 B_{02} v_2 \phi' + (T_2 B_{02})' v_2 \phi\big) + T_2 B_{02} v_2' \phi\big\}\,dh \\ &= \int_\delta^{h_2} \big\{T_2 A_2 \phi'' + T_2 B_2 v_2 \phi' - \big(T_2 B_{02} v_2 \phi' + T_2 C_{02} v_2 \phi\big) - \big(T_2 A_2 v_2'\big)'\phi - T_2 B_{02} v_2' \phi\big\}\,dh \\ &= \int_\delta^{h_2} \big\{T_2 A_2 \phi'' + T_2 B_2 v_2 \phi' - \big(T_2 B_{02} v_2 \phi' + T_2 C_{02} v_2 \phi\big) - \big(T_2 A_2 v_2'\big)'\phi - T_2 B_{02} v_2' \phi\big\}\,dh \\ &= \int_\delta^{h_2} \big\{T_2 A_2 \phi'' + T_2 B_2 v_2 \phi' - \big(T_2 B_{02} v_2 \phi' + T_2 C_{02} v_2 \phi\big) + T_2 \hat{g}_2 \phi\big\}\,dh. \end{aligned}$$

Hence,

$$\begin{aligned} &-\big(T_2 A_2 v_2 \phi' - T_2 A_2 v_2' \phi - T_2 B_{02} v_2 \phi\big)(0) \\ &\quad = \int_0^{h_2} \big\{T_2 A_2 \phi'' + T_2 B_2 v_2 \phi' - \big(T_2 B_{02} v_2 \phi' + T_2 C_{02} v_2 \phi\big) + T_2 \hat{g}_2 \phi\big\}\,dh. \end{aligned}$$

Similarly, for $i \in \{1,3\}$, we have

$$\begin{aligned}
&\big(T_i A_i v_i \phi' - T_i A_i v_i' \phi - T_i B_{0i} v_i \phi\big)(0) \\
&\quad = \int_{h_i}^{0} \big\{T_i A_i \phi'' + T_i B_i v_i \phi' - \big(T_i B_{0i} v_i \phi' + T_i C_{0i} v_i \phi\big) + T_i \hat{g}_i \phi\big\}\, dh,
\end{aligned}$$

and, therefore, in view of (3.4) and (2.8),

$$0 = \Big(-\beta_2 v_2(0) + \sum_{i=1,3} \beta_i v_i(0)\Big)\phi'(0) + \Big(\beta_2 v_2'(0) + \gamma_2 v_2(0) - \sum_{i=1,3} \big(\beta_i v_i'(0) + \gamma_i v_i(0)\big)\Big)\phi(0).$$

If we choose ϕ so that $\phi'(0) = 0$ and $\phi(0) \neq 0$ and note that $\gamma_2 = \sum_{i=1,3} \gamma_i$, by Lemma 2.5, and $v_1(0) = v_2(0) = v_3(0)$, then we find that the boundary condition

$$\beta_2 v_2'(0) = \sum_{i=1,3} \beta_i v_i'(0) \tag{3.5}$$

is satisfied. Thus the triple (v_1, v_2, v_3) is the solution of (1.16). □

We conclude this section with the

Proof of Theorem 3.1. We use a standard blow-up argument. Arguing by contradiction, we assume that there exists $\varepsilon_j \to 0$ such that

$$\lim_{j\to\infty} \|u^{\varepsilon_j}\|_{\infty,\Omega} = \infty.$$

To this end, let $M_j = \|u^{\varepsilon_j}\|_{\infty,\Omega}$, $\phi_j = u^{\varepsilon_j}/M_j$, and observe that ϕ_j is a solution of (1.4), with ε, g and ρ^ε replaced respectively by ε_j, g/M_j and ρ^{ε_j}/M_j. Moreover, as $j \to \infty$, the g/M_j's and ρ^{ε_j}/M_j's converge to zero uniformly on Ω and $\partial\Omega$ respectively. Finally, we have $\|\phi_j\|_{\infty,\Omega} = 1$ for $j \in \mathbb{N}$.

We may now apply the argument of the proof of Theorem 1.1, where the uniform boundedness of the u^ε's is assumed, to the sequence $\{\phi_j\}$ in place of $\{u^{\varepsilon_j}\}$ to conclude that the ϕ_j's converge uniformly in $\bar{\Omega}$ to $\psi_i \circ H$ on $\bar{\Omega}_i$, where the triple (ψ_1, ψ_2, ψ_3) is the unique solution of (1.16) with $d_i = 0$ for all i. Obviously, the triple $(0,0,0)$ is a solution of this ode problem. Therefore, we have $\psi_i = 0$ for all i. However, this shows that the functions ϕ_j converge to zero uniformly on Ω as $j \to \infty$, which contradicts the fact that $\|\phi_j\|_{\infty,\Omega} = 1$ for all j. □

4. The local compactness

To prove Theorem 3.2 it is necessary to obtain some, independent of ε, a priori bounds for Du^ε. Since the matrix A may be degenerate, we do not have global Lipschitz bounds for the u^ε's. To go around this difficulty, we use the structure of the Hamiltonian H. In particular we use the fact that, if for some unit vector $e \in \mathbb{R}^2$ and $x \in \Omega$, $e \cdot DH(x) \neq 0$, then, in a neighborhood of x, (1.4) behaves like a parabolic equation, with $-b$ as the time direction, and a small parameter in front of the all the other terms.

In the next theorem, we assume some a priori bounds, which we prove later, and show the existence of a convergent subsequence $u^j = u^{\varepsilon_j}$ as $\varepsilon_j \to 0$.

Theorem 4.1. *Fix $x_0 \in \Omega$ and a sequence $\varepsilon_j \to 0$, and, in addition to the assumptions of Theorem 3.2, assume that, for some unit vector $e_0 \in \mathbb{R}^2$ and a compact neighborhood $U \subset \Omega$ of x_0, $e_0 \cdot DH(x_0) \neq 0$ and*

$$\sup_{j\in\mathbb{N}}\left(\left\|u^{\varepsilon_j}\right\|_{\infty,U} + \int_U \left(e_0 \cdot Du^{\varepsilon_j}(x)\right)^2 dx\right) < \infty. \tag{4.1}$$

There exists a neighborhood V of x_0 and a subsequence $\{u^{\varepsilon_{j_k}}\}_{k\in\mathbb{N}}$ such that

$$\limsup_{k\to\infty}{}^* u^{\varepsilon_{j_k}} = \liminf_{k\to\infty}{}_* u^{\varepsilon_{j_k}} \quad \text{in } V.$$

Proof. After rotating coordinates (see Appendix A) we may assume that $e_0 = (0,1)$ and, hence, $e_0 \cdot DH(x) = H_{x_2}(x)$. Moreover to simplify the presentation we write u^j for u^{ε_j}.

First we prove the claim in the special case that, for all $x \in U$, $H(x) = x_2$ where, for $a, b > 0$, $U = [x_{01} - a, x_{01} + a] \times [x_{02} - b, x_{02} + b]$.

To this end, we write (s,t) for $x - x_0$, i.e., $x_1 = x_{01} + s$ and $x_2 = x_{02} + t$, and, thus, we regard u^j, a_{ij}, b_0 and g as functions of (s,t), and we note that, in this simplified setting, $\bar{D}H(x) = (1,0)$ for $x \in U$ and the pde for $u = u^j$ in U is

$$-\Delta_0 u - b_0 \cdot Du - \varepsilon_j^{-1} u_s = g.$$

Since, by assumption, there exists $C > 0$ such that, for all $j \in \mathbb{N}$,

$$\left\|u^j\right\|_{\infty,U} + \int_U u_t^{j\,2}\, ds\, dt \leqslant C,$$

Chebychev's inequality yields

$$\min_{a/2\leqslant s\leqslant a} \int_{-b}^{b} u_t^j(s,t)^2\, dt \leqslant \frac{2C}{a},$$

and, for each $j \in \mathbb{N}$, we may choose $s_j \in [a/2, a]$ so that

$$\int_{-b}^{b} u_t^j(s_j,t)^2\, dt \leqslant \frac{C}{a}.$$

Set, for $r \geqslant 0$, $\omega(r) = (Cr/a)^{1/2}$, and observe that, for all $t_1, t_2 \in [-b,b]$ and $j \in \mathbb{N}$,

$$\left|u^j(s_j,t_1) - u^j(s_j,t_2)\right| \leqslant \omega\left(|t_1 - t_2|\right). \tag{4.2}$$

It follows from the Ascoli–Arzela theorem that there exist $\phi \in C([-b,b])$ and a sequence $j_k \to \infty$ such that, as $k \to \infty$ and on $[-b,b]$,

$$u^{j_k}(s_j,\cdot) \to \phi(\cdot).$$

Fix $\gamma > 0$, choose $\rho \in (0, b/3)$ so that $\omega(2\rho) < \gamma$, $p, q \in C^2(\mathbb{R}; [0, \infty))$ such that

$$p = 0 \quad \text{in } [-\rho, \rho] \quad \text{and} \quad p \geqslant 2C \quad \text{in } \mathbb{R} \setminus (-2\rho, 2\rho),$$
$$q' \leqslant 0 \quad \text{in } \mathbb{R}, \qquad q = 0 \quad \text{in } [-a/2, \infty) \quad \text{and} \quad q(-a) \geqslant 2C,$$

and note that, for any $\tau \in [-b/3, b/3]$, $(-a, a) \times (\tau - 2\rho, \tau + 2\rho) \subset U$.

Finally fix $\tau \in [-b/3, b/3]$ and $j \in \mathbb{N}$, let $W = (-a, s_j) \times (\tau - 2\rho, \tau + 2\rho)$, and, for $(s, t) \in \bar{W}$, set

$$w(s, t) = u^j(s_j, \tau) + \gamma + p(t - \tau) + \gamma(s_j - s) + q(s).$$

We note that in the remainder of the proof the claims we are making are valid for sufficiently large j.

It follows that, on $\bar{W}$,

$$-w_s - \varepsilon_j(\Delta_0 w + b_0 \cdot Dw + g) \geqslant \gamma - \varepsilon_j(\Delta_0 w + b_0 \cdot Dw + g) > 0,$$

and thus, if $w \geqslant u^j$ on ∂W, the maximum principle yields

$$w \geqslant u^j \quad \text{on } W.$$

For the comparison on ∂W, observe that, if $|t - \tau| = 2\rho$, then

$$w(s, t) \geqslant u^j(s_j, \tau) + 2C \geqslant C \geqslant u^j(s_j, t),$$
$$w(-a, t) \geqslant u^j(s_j, t) + q(-a) \geqslant C \geqslant u^j(-a, t),$$

and, since $|t - \tau| \leqslant 2\rho$,

$$w(s_j, t) \geqslant u^j(s_j, \tau) + \gamma \geqslant u(s_j, \tau) + \omega(2\rho) \geqslant u^j(s_j, t).$$

Similarly, for $(s, t) \in (-a, s_j) \times (\tau - 2\rho, \tau + 2\rho)$, we get

$$u^j(s, t) \geqslant u^j(s_j, \tau) - \gamma - p(t - \tau) - \gamma(s_j - s) - q(s).$$

In particular, if $(s, t) \in (-a/2, a/2) \times (\tau - \rho, \tau + \rho)$, we have

$$\left|u^j(s, t) - u^j(s_j, \tau)\right| \leqslant \gamma(1 + s_j - s) \leqslant \gamma(1 + 2a).$$

Hence, since $\lim_{j\to\infty} u^j(s_j, \tau) = \phi(\tau)$, for $s \in (-a/2, a/2)$, we obtain

$$\limsup_{k\to\infty}{}^* u^{j_k}(s, \tau) \leqslant \phi(\tau) + 2\gamma(1 + a),$$

and

$$\liminf_{k\to\infty}{}_* u^{j_k}(s, \tau) \geqslant \phi(\tau) - 2\gamma(1 + a).$$

Finally, since $\gamma > 0$ and $\tau \in (-b/3, b/3)$ are arbitrary, we conclude that

$$\limsup_{k\to\infty}{}^* u^{j_k} = \liminf_{k\to\infty}{}_* u^{j_k} = \phi \quad \text{on } (-a/2, a/2) \times (-b/3, b/3),$$

and the proof of the claim in this simplified setting is complete.

Next, we show that it is possible, after a change of variables, to transform the general setting into the one studied above.

To this end, let $\Phi : U \to \mathbb{R}^2$ be given by $\Phi(x) = (x_1, H(x))$. Since $H_{x_2}(x_0) > 0$, Φ is an order-preserving diffeomorphism from a neighborhood of x_0 to a neighborhood of $(x_{01}, H(x_0))$. Setting $v^j(\Phi(x)) = u^j(x)$ and $\tilde{H}(\Phi(x)) = H(x)$, in the new variable $y = \Phi(x)$, we have

$$\tilde{H}(y) = y_2 \quad \text{and} \quad \bar{D}\tilde{H}(y) = (1, 0).$$

Consequently, in view of the invariance of the form of the pde (1.4) under change of variables (see Appendix A), we deduce that, in a neighborhood of $(x_{01}, H(x_0))$ and for some $\tilde{b}_0$ and $\tilde{g}$,

$$-\tilde{\Delta}_0 v^j - \tilde{b}_0 \cdot Dv^j - \varepsilon_j^{-1} v^j_{y_1} = \tilde{g},$$

where $\tilde{\Delta}_0 w = (\tilde{a}_{11} w_{y_1} + \tilde{a}_{12} w_{y_2})_{y_1} + (\tilde{a}_{21} w_{y_1} + \tilde{a}_{22} w_{y_2})_{y_2}$ for some $\tilde{a}_{ij}$.

Also, noting that $u^j_{x_2}(x) = v^j_{y_2}(\Phi(x)) H_{x_2}(x)$ and $\det D\Phi(x) = H_{x_2}(x)$, we find that, in a small neighborhood $\tilde{U}$ of $(x_{01}, H(x_0))$,

$$\sup_{j\in\mathbb{N}} \left(\|v^j\|_{\infty,\tilde{U}} + \int_{\tilde{U}} v^j_{y_2}(y)^2 \, dy \right) < \infty.$$

The proof is now complete. □

We proceed with the proofs of the (4.1) and, in particular, the integral bound since the sup-estimate follows from Theorem 3.1. Throughout the arguments below we assume that Theorem 3.4 holds. Its proof will be presented in Section 6.

We have:

Lemma 4.1. *Let u^ε be a solution of* (1.4)*. For any compact subset K of Ω, there exists a constant $C_K > 0$ such that, for all $0 < \varepsilon < 1$,*

$$\int_K |D_0 u^\varepsilon|^2 \, dx \leqslant C_K \left(\|u^\varepsilon\|^2_{\infty,\Omega} + 1 \right).$$

Proof. Fix $\varepsilon \in (0, \varepsilon_0)$ and let $\phi \in C^2_0(J_i)$. Then

$$\int_{\Omega_i} \left\{ \left(\Delta_0 + \left(b_0 + \varepsilon^{-1} b \right) \cdot D \right) u^\varepsilon + g \right\} \phi \circ H \, dx = 0.$$

In addition,

$$\begin{aligned}
\int_{\Omega_i} \Delta_0 u^\varepsilon u^\varepsilon \phi \circ H \, dx &= - \int_{\Omega_i} \left(|D_0 u^\varepsilon|^2 \phi \circ H + \phi' \circ H \langle Du^\varepsilon, Dh \rangle_0 \right) dx \\
&\geqslant - \int_{\Omega_i} \left(|D_0 u^\varepsilon|^2 \phi \circ H - |D_0 u^\varepsilon| |D_0 H| |\phi' \circ H| \right) dx,
\end{aligned}$$

where, for $x \in \Omega$ and $\xi, \eta \in \mathbb{R}^2$,

$$\langle \xi, \eta \rangle_0 = a_{11}(x)\xi_1\eta_1 + a_{12}(x)(\xi_1\eta_2 + \xi_2\eta_1) + a_{22}(x)\xi_2\eta_2.$$

Moreover,

$$2\int_{\Omega_i} (b_0 \cdot Du^\varepsilon)u^\varepsilon \phi \circ H\,dx = \int_{\Omega_i} (\phi \circ Hb_0) \cdot D(u^\varepsilon)^2\,dx = -\int_{\Omega_i} (u^\varepsilon)^2 \operatorname{div}(\phi \circ Hb_0)\,dx,$$

and

$$2\int_{\Omega_i} (b \cdot Du^\varepsilon)u^\varepsilon \phi \circ H\,dx = -\int_{\Omega_i} (u^\varepsilon)^2 \operatorname{div}(\phi \circ Hb)\,dx = -\int_{\Omega_i} (u^\varepsilon)^2 \phi' \circ Hb \cdot DH\,dx = 0.$$

Replacing ϕ by ϕ^2 in the above computation and recalling Theorem 3.1, we find $C > 0$, depending only on $\sup_{0<\varepsilon<\varepsilon_0} \|u^\varepsilon\|_{\infty,\Omega}$, H, ϕ, a_{ij} and b_0 such that

$$\int_{\Omega_i} |D_0 u^\varepsilon|^2 \phi^2 \circ H\,dx \leqslant C\big(\|u^\varepsilon\|_{\infty,\Omega}^2 + 1\big). \tag{4.3}$$

Similarly, if $\phi \in C_0^2(J)$ with $J = (\max_{i=1,3} h_i, h_2)$, then, for some $C > 0$, we also have

$$\int_{\Omega} |D_0 u^\varepsilon|^2 \phi^2 \circ H\,dx \leqslant C\big(\|u^\varepsilon\|_{\infty,\Omega}^2 + 1\big). \tag{4.4}$$

Combining (4.3) and (4.4) yields the claim. □

We present now the

Proof of Theorem 3.2. The precompactness of the family $\{u^\varepsilon\}_{\varepsilon>0}$ in $C(N)$ follows from standard compactness and diagonal arguments once we show that, for each $x_0 \in N$ and every sequence $\varepsilon_j \to 0$, there exists a subsequence $\varepsilon_{j_k} \to 0$ and a neighborhood V of x_0 such that, if $u^k = u^{\varepsilon_{j_k}}$,

$$\limsup_{k\to\infty}{}^* u^k = \liminf_{k\to\infty}{}_* u^k \quad \text{in } V.$$

To prove the claim we use Theorem 4.1. Thus we only need to find a vector $e_0 \in \mathbb{R}^2$ for which (4.1) holds. This will be done again using a convenient change of variables.

To simplify the notation, for the rest of the proof, we write $\alpha = a_{11}$, $\beta = a_{12} = a_{21}$ and $\gamma = a_{22}$ suppressing, unless necessary, the explicit x-dependence, and, after some relabeling, we assume that $j = 1$.

Fix a compact neighborhood $U \subset \Omega$ of x_0. It follows from Theorem 3.1 and Lemma 4.1, that, for $\tilde{j} \in \mathbb{N}$ large enough,

$$\sup_{j\geqslant\tilde{j}}\Big(\|u^j\|_{\infty,U} + \int_U |D_0 u^j|^2\,dx\Big) < \infty.$$

Since $ADH \cdot DH(x_0) > 0$, we have either $(\alpha, \beta) \cdot DH(x_0) \neq 0$ or $(\beta, \gamma) \cdot DH(x_0) \neq 0$, since, otherwise, $ADH(x_0) = 0$. Next we assume $(\beta, \gamma) \cdot DH(x_0) \neq 0$. The other case can be treated in a similar way.

Set $e = (\beta, \gamma)$ on Ω. By replacing, if needed, U by a smaller neighborhood, we may assume that $e \cdot DH \neq 0$ in U. The degenerate ellipticity of Δ_0 yields $\alpha\gamma \geqslant \beta^2$ in Ω and, therefore, we must have $\gamma > 0$ in U.

Observe that, for any $y = (y_1, y_2) \in \mathbb{R}^2$, $x \in U$ and $C_\gamma = \max_U \gamma$,

$$\begin{aligned}\big(e(x) \cdot y\big)^2 &= \beta^2(x) y_1^2 + 2\beta(x)\gamma(x) y_1 y_2 + \gamma^2(x) y_2^2 \\ &\leqslant \alpha(x)\gamma(x) y_1^2 + 2\beta(x)\gamma(x) y_1 y_2 + \gamma^2(x) y_2^2 = \gamma(x) A(x) y \cdot y \leqslant C_\gamma A(x) y \cdot y.\end{aligned}$$

Hence,

$$\int_U \big(e(x) \cdot Du^j(x)\big)^2\, dx \leqslant C_\gamma \int_U \big|D_0 u^j\big|^2\, dx.$$

Next we change variables to “straighten” the vector field e. To this end, let $X(s,t)$ be the solution of the initial value problem

$$\frac{\partial X(s,t)}{\partial t} = e\big(X(s,t)\big) \quad \text{with } X(s,0) = x_0 + (s,0),$$

and recall that $X(s,t)$ is smooth in a neighborhood W of the origin $(s,t) = (0,0)$.

Note that, since

$$DX = \begin{pmatrix} X_{1,s} & \beta(X) \\ X_{2,s} & \gamma(X) \end{pmatrix},$$

where $X = (X_1, X_2)$ and $X_{1,s}(s,0) = 1$, it follows that $\det DX(0,0) = \gamma(x_0) > 0$. Thus, by reselecting, if necessary, W small enough and setting $U = X(W)$, since $\gamma(x_0) > 0$, we may assume that $X : W \to U$ is an orientation-preserving diffeomorphism.

Let $x = X(y)$. Setting $v^j(y) = u^j(X(y))$ and $e_2 = (0,1) \in \mathbb{R}^2$ and noting that $DX(y)e_2 = e(X(y))$, we get, for $C_X = 1/\min_W |\det X|$,

$$\begin{aligned}\int_W \big(e_2 \cdot Dv^j(y)\big)^2\, dy &= \int_W \big(e_2 \cdot DX(y)^* Du^j\big(X(y)\big)\big)^2 dy \\ &\leqslant C_X \int_W \big(DX(y)e_2 \cdot Du^j\big(X(y)\big)\big)^2 \big|\det X(y)\big|\, dy \\ &\leqslant C_X \int_W \big(e \cdot Du^j\big)^2 \circ X\, dx,\end{aligned}$$

where $DX(y)^*$ denotes the transposed matrix of $DX(y)$.

It follows that (4.1) holds with $e_0 = e_2$ in the new coordinate system. Similarly, if we set $\tilde{H} = H \circ X$, then

$$e_2 \cdot D\tilde{H}(y) = e_2 \cdot DX(y)^* DH \circ X = e \cdot DH \circ X \neq 0.$$

Applying Theorem 4.1 we conclude the proof. $\square$

5. The construction of the barriers

The key step of the proof of Theorem 3.3 is

Theorem 5.1. *Let h_0 and I_i be as in Theorem* 3.3*, set $W_i = \Omega_i \cap \{|H| > h_0\}$, and assume that $w_i \in C^4(I_i)$ satisfy*

$$-\big(A_i w_i'' + (B_i + B_{0i}) w_i'\big) + 2 \leqslant 0 \quad in\ I_i.$$

There exist $\zeta_i^\varepsilon \in C^2(\bar{W}_i)$ and $\varepsilon_0 > 0$ such that, if $\varepsilon \in (0, \varepsilon_0)$, then

$$-\big(\Delta_0 + \big(b_0 + \varepsilon^{-1} b\big) \cdot D\big)\zeta_i^\varepsilon + 1 \leqslant 0 \quad in\ W_i \quad and \quad \lim_{\varepsilon \to 0} \big\| w_i - \zeta_i^\varepsilon \big\|_{\infty, W_i} = 0.$$

Before going into the proof of Theorem 5.1, we need to introduce some auxiliary functions. To this end, recall that l_1 and l_3 are respectively the curves $\{Y_1(h)\colon\ h_1 \leqslant h \leqslant h_2\}$ and $\{Y_3(h)\colon\ h_3 \leqslant h \leqslant h_2\}$, while l_2 is either of l_1 and l_3. Then for each $x \in V_i = \Omega_i \cup \partial_{\mathrm{out}} \Omega_i$, $\tau_i(x)$ is the first time the flow $(X(t,x))_{t>0}$ hits the curve l_i, i.e.,

$$X\big(\tau_i(x), x\big) \in l_i \quad \text{and} \quad X(t,x) \notin l_i \quad \text{for all } t \in \big(0, \tau_i(x)\big).$$

It follows that $\tau_i = T_i \circ H$ in $l_i \cap V_i$ and $\tau_i \leqslant T_i \circ H$ in V_i. Also note that, although τ_i is continuous in $V_i \setminus l_i$, it is not, in general, continuous across l_i. To go around this difficulty, we modify the τ_i's near l_i by considering the neighborhoods $U_i = \{x \in V_i\colon\ \tau_i(x) \neq T_i(H(x))/2\}$ and the continuous maps $\tilde{\tau}_i : U_i \to (0, \infty)$ defined by

$$\tilde{\tau}_i(x) = \begin{cases} \tau_i(x) & \text{if } \tau_i(x) > T_i(H(x))/2, \\ \tau_i(x) + T_i(H(x)) & \text{if } \tau_i(x) < T_i(H(x))/2. \end{cases}$$

When we discuss the regularity of the τ_i's near $l_i \cap V_i$, we implicitly refer to the $\tilde{\tau}_i$'s.

We have:

Lemma 5.1. $\tau_i \in C^3(V_i)$.

Proof. As it has been already noted in the proof of Lemma 2.1, there exists $\phi \in C^3(\mathbb{R}^2; \mathbb{R})$ such that $\phi = 0$ and $D\phi \neq 0$ on l_i. Set $\psi(t,x) = \phi(X(t,x))$. Since $\psi(\tau_i(x), x) = 0$ for $x \in V_i$ and $\psi_t(\tau_i(x), x) = D\phi(Y_i(H(x))) \cdot b(Y_i(H(x))) \neq 0$ for $x \in V_i$, the implicit function theorem yields that τ_i is locally (in the sense that it has to be replaced by $\tilde{\tau}_i$ near l_i) of class C^3. $\square$

We continue with the

Proof of Theorem 5.1. We only show the existence of ζ_2^ε, since the construction of ζ_1^ε and ζ_3^ε is similar.

To this end, recall that $X, \dot{X} \in C^2(\mathbb{R} \times \bar{W}_2)$, $\tau \in C^2(\bar{W}_2)$, $T_2 \in C^2(\bar{I}_2)$, $A_2, B_2 \in C^2(\bar{I}_2)$, $w_2 \in C^4(\bar{I}_2)$, set

$$f = (\Delta_0 + b_0 \cdot D)(w_2 \circ H) - \big(A_2 w_2'' \circ H + (B_2 + B_{02}) w'\big) \circ H,$$

and observe that, if $h = H(x)$ for $x \in W_2$, then

$$f(x) = w_2''(h)\big|D_0 H(x)\big|^2 + w_2'(h)\big(\Delta_0 H(x) + b_0(x) \cdot DH(x)\big) - \big(A_2 w_2'' + (B_2 + B_{02}) w_2'\big)(h),$$

and

$$\int_0^{T_2(h)} f\big(X(t,x)\big)\,dt = T_2\big(A_2 w_2'' + (B_2 + B_{02}) w_2'\big) - T_2\big(A_2 w_2'' + (B_2 + B_{02}) w_2'\big) = 0. \tag{5.1}$$

For $x \in \bar{W}_2$, define

$$\chi(x) = \int_0^{\tau(x)} f\big(X(t,x)\big)\,dt.$$

It is clear that $\chi \in C^2(\bar{W}_2 \setminus l_1)$. Moreover, recalling the notation $\tilde{\tau}$ and U and the fact that either $\tilde{\tau} = \tau$ or $\tilde{\tau} = \tau + T_2 \circ H$ in $\bar{U} \cap \bar{W}_2$, we obtain from (5.1) that, for any $x \in U \cap \bar{W}_2$,

$$\chi(x) = \int_0^{\tilde{\tau}(x)} f\big(X(t,x)\big)\,dt,$$

and, hence, $\chi \in C^2(U \cap \bar{W}_2)$ and $\chi \in C^2(\bar{W}_2)$.
It turns out that χ is a solution of

$$-b \cdot D\chi = f \quad \text{in } W_2. \tag{5.2}$$

Indeed fix any $x \in W_2 \setminus l_1$ and observe that, if $t > 0$ is sufficiently small,

$$\tau(x) = \tau\big(X(t,x)\big) + t.$$

Then

$$\begin{aligned}
b(x) \cdot D\chi(x) &= \frac{\partial}{\partial t}\chi\big(X(t,x)\big)\Big|_{t=0} = \frac{\partial}{\partial t} \int_0^{\tau(X(t,x))} f\big(X\big(s, X(t,x)\big)\big)\,ds\Big|_{t=0} \\
&= \frac{\partial}{\partial t} \int_0^{\tau(x)-t} f\big(X(s+t,x)\big)\,ds\Big|_{t=0} \\
&= -f\big(X\big(\tau(x),x\big)\big) + \int_0^{\tau(x)} \frac{\partial}{\partial s} f\big(X(s,x)\big)\,ds \\
&= -f\big(X\big(\tau(x),x\big)\big) + f\big(X\big(\tau(x),x\big)\big) - f\big(X(0,x)\big) = -f(x),
\end{aligned}$$

i.e., χ satisfies (5.2) in $W_2 \setminus l_1$ and, since $\chi \in C^2(W_2)$, it satisfies (5.2) in W_2.
Finally, we define $\zeta_2^\varepsilon \in C^2(\bar{W}_2)$ by $\zeta_2^\varepsilon = w_2 \circ H + \varepsilon\chi$. If $u = \zeta_2^\varepsilon$, then, for some $C > 0$,

$$\begin{aligned}
-\big(\Delta_0 + \big(b_0 + \varepsilon^{-1} b\big) \cdot D\big)u &= -(\Delta_0 + b_0 \cdot D)(w_2 \circ H) - b \cdot D\chi - \varepsilon(\Delta_0 + b_0 \cdot D)\chi \\
&= -(\Delta_0 + b_0 \cdot D)(w_2 \circ H) + f - \varepsilon(\Delta_0 + b_0 \cdot D)\chi
\end{aligned}$$

$$= -\big(A_2 w_2'' + (B_2 + B_{02}) w_2'\big) \circ H - \varepsilon(\Delta_0 + b_0 \cdot D)\chi$$
$$\leqslant -2 - \varepsilon(\Delta_0 + b_0 \cdot D)\chi \leqslant -2 + C\varepsilon.$$

Since $\varepsilon \in (0, \varepsilon_0)$ and ε_0 is sufficiently small, we may assume that

$$-\big(\Delta_0 + \big(b_0 + \varepsilon^{-1} b\big) \cdot D\big) u \leqslant -1 \quad \text{in } W_2.$$

The uniform convergence of the ζ_2^ε's to $w_2 \circ H$ in W_2 is obvious. $\square$

We present here the

Proof of Theorem 3.3. We only discuss the case $i = 2$, since the arguments for $i = 1, 3$ are similar.
Choose $w \in C^4(\bar{I}_2)$ (for instance, a quadratic function) such that

$$-\big(A_2 w'' + B_2 w'\big) \leqslant -2 \quad \text{in } I_2, \qquad w(h_0) \leqslant -2 \quad \text{and} \quad w(h_2) = d_2.$$

Using Theorem 5.1 we find a family $\{w^\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)} \subset C^2(\bar{W}_2)$ such that

$$-\big(\Delta_0 + \varepsilon^{-1} b \cdot D\big) w^\varepsilon \leqslant -1 \quad \text{in } W_2 \quad \text{and} \quad \lim_{\varepsilon \to 0+} \big\| w^\varepsilon - v \big\|_{\infty, W_2} = 0.$$

A minor modification of the w^ε's yields a desired family $\{w_2^\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)}$. $\square$

The rest of the section is devoted to the

Proof of Theorem 3.4. In view of (1.9) and (1.10), we may assume, by choosing $\kappa > 0$ small enough, that $a_{11} > 0$ in S_κ.

As in [6], we consider the function $E \in C^\infty(\mathbb{R})$ given by

$$E(x) = \int_0^x e^{-t^2} \int_0^t e^{s^2}\, ds\, dt,$$

and note that, for $x \in \mathbb{R}$,

$$E(-x) = E(x), \qquad E''(x) + 2xE'(x) = 1, \qquad E(0) = E'(0) = 0,$$

and

$$\lim_{x \to \infty} \frac{E(x)}{\log x} = \lim_{x \to \infty} xE'(x) = \frac{1}{2}.$$

Accordingly, we can choose $C_0 > 0$ such that

$$0 \leqslant E(x) \leqslant C_0 \log(x + 2) \quad \text{and} \quad 0 \leqslant E'(x) \leqslant C_0 \quad \text{in } [0, \infty),$$

and, for all $x \in \mathbb{R}$,

$$E''(x) + 2xE'(x) - \frac{1}{2C_0}\big|E'(x)\big| \geqslant \frac{1}{2}.$$

We define the function v on S_κ (or on the interval $[-\kappa,\kappa]$) by

$$v(x) = v(x_1) = \alpha E(\beta x_1),$$

where $\alpha > 0$ and $\beta > 0$ are constants to be fixed later.

Next choose constants $\mu > 0$ and $\theta > 0$ such that $\mu \geqslant \|a_{11}\|_{\infty,\Omega} + \|Da_{11}\|_{\infty,\Omega} + \|Da_{12}\|_{\infty,\Omega} + \|b_0\|_{\infty,\Omega}$ and $a_{11} \geqslant \theta$ in S_κ.

A straightforward calculation yields that, as function of x_1, v satisfies

$$v'' + 2\beta^2 x_1 v' - \frac{\beta}{2C_0}|v'| \geqslant \frac{\alpha\beta^2}{2} \quad \text{in } [-\kappa,\kappa],$$

while, as a function on S_κ,

$$\begin{aligned}
&\big(\Delta_0 + \big(b_0 + \varepsilon^{-1} b\big)\cdot D\big)v \\
&= a_{11}(x)v''(x_1) + \big(a_{11,x_1}(x) + a_{21,x_2}(x)\big)v'(x_1) + b_{01}(x)v'(x_1) + \frac{1}{\varepsilon}x_1 v'(x_1) \\
&\geqslant a_{11}\bigg(\frac{\alpha\beta^2}{2} - 2\beta^2 x_1 v' + \frac{\beta}{2C_0}|v'|\bigg) - \mu|v'| + \frac{1}{\varepsilon}x_1 v'(x) \\
&= \frac{\theta\alpha\beta^2}{2} + \bigg(\frac{1}{\varepsilon} - 2\mu\beta^2\bigg)x_1 v' + \bigg(\frac{\theta\beta}{2C_0} - \mu\bigg)|v'|.
\end{aligned}$$

Next we fix α, β so that

$$\theta\alpha\beta^2 \geqslant 2, \qquad 1 \geqslant 2\mu\beta^2\varepsilon \quad \text{and} \quad \theta\beta \geqslant 2C_0\mu. \tag{5.3}$$

Indeed set

$$\alpha = 4\mu\varepsilon\theta^{-1} \quad \text{and} \quad \beta = (2\mu\varepsilon)^{-1/2}.$$

It follows that (5.3) is satisfied for $\varepsilon \in (0,\varepsilon_0)$, provided that $\varepsilon_0 \in (0,1)$ is so small that

$$\frac{\theta\beta}{2C_0} = \frac{\theta}{2C_0\sqrt{2\mu\varepsilon}} \geqslant \mu.$$

We write v_ε for v and note that

$$-\big(\Delta_0 + \big(b_0 + \varepsilon^{-1} b\big)\cdot D\big)v_\varepsilon \leqslant -1 \quad \text{in } S_\kappa.$$

Also, since

$$v_\varepsilon(x) = \frac{4\mu\varepsilon}{\theta}E\bigg(\frac{1}{\sqrt{2\mu\varepsilon}}x_1\bigg),$$

we find that, for some $C > 0$ independent of ε,

$$0 \leqslant v_\varepsilon(x) \leqslant C\varepsilon\log\big(\varepsilon^{-1} + 2\big).$$

The family $\{v_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ has the required properties. $\square$

Appendix A. Invariance under change of variables

For the convenience of the reader we record here the change of variable formula for the pde in (1.4).

Let $x = \Phi(y)$ be a C^2 diffeomorphism from $U \subset \mathbb{R}^2$ to $V \subset \Omega$. Assume that $u \in C^2(V)$ satisfies the pde in (1.4) for some $\varepsilon > 0$. Then $\tilde{u}(y) = u \circ \Phi$ satisfies the pde

$$-\operatorname{div}(\tilde{A} D\tilde{u}) - \tilde{b}_0 \cdot D\tilde{u} - \varepsilon^{-1} \bar{D}\tilde{H} \cdot D\tilde{u} = \tilde{g} \quad \text{in } U,$$

where

$$\begin{aligned}
\tilde{A}(y) &= \det D\Phi(y) D\Phi(y)^{-1} A \circ \Phi(y) \bigl(D\Phi(y)^{-1}\bigr)^*, \\
\tilde{b}_0(y) &= \det D\Phi(y) D\Phi(y)^{-1} b_0 \circ \Phi(y), \\
\tilde{H}(y) &= H \circ \Phi(y), \\
\tilde{g}(y) &= \det D\phi(y) g \circ \Phi(y).
\end{aligned}$$

This can be checked by a direct computation, which we leave to the interested reader. If Φ is orientation-preserving, i.e., $\det \Phi > 0$, then $\tilde{A}(y)$ is nonnegative definite. Otherwise, it is nonpositive definite, and, to keep the structure of degenerate ellipticity of the pde, one has to multiply $(\tilde{A}, \tilde{b}_0, \tilde{H}, \tilde{g})$ by a negative constant (e.g., -1), which introduces a change of the sign of the Hamiltonian.

Appendix B. Green's formula

We state and prove here a simple consequence of Green's formula, which is used repeatedly in the paper.

Lemma B.1. *Let $f = (f_1, f_2) \in C^1(\Omega)$ and $\alpha_i \in J_i$, with $i = 1, 2, 3$. Then*

$$\sum_{i=1}^{3} (-1)^i \int_0^{T_i(\alpha_i)} (f \cdot DH) \circ \Phi_i(t, \alpha_i)\, dt = \sum_{i=1}^{3} (-1)^i \int_0^{\alpha_i} \int_0^{T_i(h)} \operatorname{div} f \circ \Phi_i(t, h)\, dt\, dh.$$

Proof. If $U = \{H = 0\} \cup \{x \in \Omega_2\colon\ H(x) < \alpha_2\} \cup \bigcup_{i=1,3} \{x \in \Omega_i\colon\ H(x) > \alpha_i\}$, Green's formula yields

$$\int_{\partial U} f \cdot \nu\, dl = \int_U \operatorname{div} f\, dx,$$

where ν and dl denote respectively the outer unit vector on ∂U and the line element on ∂U. Note that $\partial U = \bigcup_{i=1}^3 c_i(\alpha_i)$, $\nu = DH/|DH|$ on $c_2(\alpha_2)$ and $\nu = -DH/|DH|$ on $\bigcup_{i=1,3} c_i(\alpha_i)$ and, if the loops $c_i(\alpha_i)$ are parametrized by $x = \Phi_i(t, \alpha_i)$ with $t \in [0, T_i(\alpha_i))$, then $dl = |DH(\Phi_i(t, \alpha_i))|\, dt$. Thus,

$$\int_{\partial U} f \cdot \nu\, dl = \sum_{i=1}^{3} (-1)^i \int_0^{T_i(\alpha_i)} (f_1 H_{x_1} + f_2 H_{x_2}) \circ \Phi_i(t, \alpha_i)\, dt.$$

On the other hand, recalling (1.15), we find

$$\int_U \operatorname{div} f\,dx = \sum_{i=1}^{3}(-1)^i \int_0^{\alpha_i} dh \int_0^{T_i(\alpha_i)} (\operatorname{div} f) \circ \Phi_i(t,h)\,dt.$$

Combining these observations completes the proof. □

References

[1] Lawrence Conlon, Differentiable Manifolds, second ed., Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2008.

[2] Michael G. Crandall, Hitoshi Ishii, Pierre-Louis Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. (N.S.) 27 (1) (1992) 1–67.

[3] Mark Freidlin, Matthias Weber, Random perturbations of nonlinear oscillators, Ann. Probab. 26 (3) (1998) 925–967.

[4] Mark Freidlin, Matthias Weber, On random perturbations of Hamiltonian systems with many degrees of freedom, Stochastic Process. Appl. 94 (2) (2001) 199–239.

[5] Mark Freidlin, Matthias Weber, Random perturbations of dynamical systems and diffusion processes with conservation laws, Probab. Theory Related Fields 128 (3) (2004) 441–466.

[6] Mark I. Freidlin, Alexander D. Wentzell, Random perturbations of Hamiltonian systems, Mem. Amer. Math. Soc. 109 (523) (1994), viii+82.

[7] Richard B. Sowers, Stochastic averaging near homoclinic orbits via singular perturbations, in: IUTAM Symposium on Nonlinear Stochastic Dynamics, in: Solid Mech. Appl., vol. 110, Kluwer Acad. Publ., Dordrecht, 2003, pp. 83–94.

[8] Richard B. Sowers, Random perturbations of two-dimensional pseudoperiodic flows, Illinois J. Math. 50 (1–4) (2006) 853–959 (electronic).